Semi-deterministic Sparse Matrix for Low Complexity Compressive Sampling

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Received August 27, 2016; revised November 7, 2016; accepted February 3, 2017; published May 31, 2017

Abstract

The construction of completely random sensing matrices of Compressive Sensing requires a large number of random numbers while that of deterministic sensing operators often needs complex mathematical operations. Thus both of them have difficulty in acquiring large signals efficiently. This paper focuses on the enhancement of the practicability of the structurally random matrices and proposes a semi-deterministic sensing matrix called Partial Kronecker product of Identity and Hadamard (PKIH) matrix. The proposed matrix can be viewed as a sub matrix of a well-structured, sparse, and orthogonal matrix. Only the row index is selected at random and the positions of the entries of each row are determined by a deterministic sequence. Therefore, the PKIH significantly decreases the requirement of random numbers, which has a complex generating algorithm, in matrix construction and further reduces the complexity of sampling. Besides, in order to process large signals, the corresponding fast sampling algorithm is developed, which can be easily parallelized and realized in hardware. Simulation results illustrate that the proposed sensing matrix maintains almost the same performance but with at least 50% less random numbers comparing with the popular sampling matrices. Meanwhile, it saved roughly 15%-35% processing time in comparison to that of the SRM matrices.

Keywords: Compressive sensing, Sparse Matrix, Semi-deterministic, Low Complexity, Kronecker product

This work was supported by the National Natural Science Foundation of China under Grant No. 61372069 and No. 61431010, the National Basic Research Program of China under Grant No. 2014CB340204, National Defense Pre-research Foundation, SRF for ROCS, SEM (JY0600090102), "111" project (No. B08038), and the Fundamental Research Funds for the Central Universities.

https://doi.org/10.3837/tiis.2017.05.009 **ISSN : 1976-7277**

1. Introduction

The design of the sensing matrix $\mathbf{\Phi} \in \mathbb{R}^{M \times N}$ $(M << N)$ is one of the three key problems of Compressive Sensing [1] (or Compressed Sampling, CS), since the sensing matrix determines whether the original signal $\mathbf{x} \in \mathbb{R}^N$ could be recovered from the incomplete projections $\mathbf{y} \in \mathbb{R}^N$ by $\mathbf{y} = \mathbf{\Phi x}$, where x is a k-sparse vector who has only k nonzero entries. Besides, the structure and the values of $\mathbf{\Phi}$ greatly affect the complexity of the applications of CS. Therefore, a "good" sensing matrix may have good sensing performance and low complexity simultaneously. In detail, the sensing matrix is able to recover any k-sparse signal when m is on the order of $O(k \log N)$. Meanwhile, $\mathbf{\Phi}$ should be easily stored and proceeded, which is more sensitive (very important) in practical applications. It has been proved that Gaussian random matrices [4], [6] and Bernoulli random matrices [6] have good performance. However, they inherently have drawbacks like huge storage requirement and high computational complexity.

To overcome these two drawbacks under the promise of good performance, various sensing matrices have been investigated. Taking advantages of fast computing methods like Fast Fourier Transform or Fast Hadamard Transform, the partial Fourier matrix [7] and the partial Hadamard matrix [7], [8] have made it possible to apply in practice. But they are random, dense and complex valued, thus are still hard to implement. On the other hand, aims to remove the randomness and makes it easier for hardware implementation, deterministic matrices [8], [9] are studied. Making uses of the popular codes of channel coding and sequences with certain characteristics, deterministic matrices usually have good sensing performance. Unfortunately, deterministic matrices always need a lot of complex mathematical operations during construction thus have difficulty in acquiring large signals. Besieds, because of dense matrix is not suitable for processing, the sparse version of random and deterministic sensing matrices have also been investigated [10], [11]. They are helpful of reducing processing complexity but can not overcome strutral or design shortage.

In addition, structured compressive sensing matrices like Toeplitz and Circulant matrices which arise from realistic scenarios are investigated [12]-[17]. These matrices are highly structured, thus can be stored efficiently. Among them, the Structured Random Matrix [23] (SRM) offers high sparsity, low complexity and fast computation properties and is universal for a variety of sparse signals. But it still uses lot random numbers during matrix construction, which makes it hard to implement because the random numbers generation is hard and time consuming.

Nevertheless, compared with the completely random matrices, the structured matrices [18]-[23] imply that, in a scene, well designed matrix structure and randomness of matrix entries are replaceable in terms of the sensing performance.

With this idea, under promise of little performance loss, we focus on to design a well-structured sensing matrix to further reduce the implementation complexity of sensing matrix by decreasing the randomness and lowering the sensing complexity of the matrix.

In this paper, we propose a semi-deterministic sparse matrix called Partial Kronecker product of Identity and Hadamard (PKIH) matrix. We fix the sampling positions using a deterministic sequence and choose rows from the core matrix uniformly at random to generate measurements. And the core matrix is designed as an orthogonal matrix consisting of a two-layer structure. The outer layer is designed as block diagonal in order to reduce the sampling delay and memory consuming, and the inner layer is designed as the Kronecker product of the Hadamard and the Identity matrix to seek for simplicity and orthogonal property.

The proposed PKIH matrix needs few random numbers and provides properties of sparse, highly structured and fast sampling. Besides, it can be easily implemented on hardware. The PKIH matrix offers comparable sensing performance to the optimal sensing matrices with much less random numbers. Moreover, the corresponding fast sampling algorithm for the acquirement of large signals is proposed, which reduces the computational complexity of the sampling procedure.

The remainder of this paper is organized as follows: In section II, we briefly introduce the background of compressive sensing and some notations. In section III, the main concept of PKIH, including matrix structure, matrix property, and the fast sensing procedure are described in detail. Section IV gives several experiments to evaluate the performance and the complexity of PKIH. Finally, section V comes to the conclusion.

2. Preliminaries and Background

2.1 Preliminaries and Notations

We use boldface letters to denote vectors (lowercase) and matrices (capital), and calligraphy letters to denote sets. The set of $\{1, 2, \dots, N\}$ is denoted by $[N]$. The entry in the *i*-th row and *j*-th column of a matrix A is denoted as a_{ij}. The matrix $\mathbf{A}$ of size $N \times N$ is denoted as $\mathbf{A}_N$ and $\mathbf{A}_N^{\mathfrak{X}}$ denotes the sub matrix of $\mathbf{A}_N$ consisting of the rows indexed by set $\mathfrak{X} \subset [N]$. $\|\cdot\|_p$ denotes the ℓ_p norm. $(\cdot)^T$ denotes the matrix transpose and $|\mathfrak{X}|$ denotes the cardinality of set $\mathfrak{X}$ .The Kronecker product of matrices $A \in \mathbb{R}^{I\times J}$ and $B \in \mathbb{R}^{K\times L}$ is denoted by $A \otimes B$. The result is a matrix of size $(I \cdot K) \times (J \cdot L)$ defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1J}B \\ a_{21}B & a_{22}B & \cdots & a_{2J}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{I1}B & a_{I2}B & \cdots & a_{IJ}B \end{pmatrix} \tag{1}$$

The transpose property and the mixed-product property of it can be expressed as

$$\begin{aligned} (\mathbf{A} \otimes \mathbf{B})^T &= \mathbf{A}^T \otimes \mathbf{B}^T \\ (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) &= (\mathbf{AC}) \otimes (\mathbf{BD}) \end{aligned} \tag{2}$$

2.2 Background of Compressive Sensing

Compressive sensing provides a new paradigm for signal acquisition and processing. The theory of CS has established that a sparse or compressible signal can be recovered with high probability from a few measurements, which is far smaller than the length of the original signal.

Consider an *n*-dimensional signal vector $\mathbf{x} = (x_1, \ldots x_N)^T$. The vector is *k*-sparse if it has at most $k\,(k << N)$ large coefficients while the remaining coefficients are small or zero. Furthermore, suppose that an *n*-dimensional signal vector $\mathbf{f} = (f_1, \ldots, f_N)^T$ can be represented as $\mathbf{f} = \sum_{i=1}^{n} x_i \boldsymbol{\psi}_i = \boldsymbol{\Psi}\mathbf{x}$ in some domain $\boldsymbol{\Psi} = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_N)$, it is said that $\mathbf{f}$ is compressible.

The theory of CS states that, with high probability, the *k*-sparse vector $\mathbf{x}$ (and further, $\mathbf{f}$) can be recovered from $M\,(M << N)$ linear combinations of measurements. It can be obtained as follows, given measurement matrix $\boldsymbol{\Phi} \in \mathbb{R}^{M \times N}$ and the observation vector $\mathbf{y} \in \mathbb{R}^M$ with

$$\mathbf{y} = \boldsymbol{\Phi}\mathbf{f} = \boldsymbol{\Phi}\boldsymbol{\Psi}\mathbf{x} \tag{3}$$

Recoverment of $\mathbf{x} \in \mathbb{R}^N$ or $\mathbf{f}$ from $\mathbf{y}$ can be achieved through solving the NP-hard ℓ_0-minimization problem

$$(P_0) \quad \min \|\mathbf{x}\|_{\ell_0} \quad s.t.\, \mathbf{y} = \boldsymbol{\Phi}\mathbf{f},\, \mathbf{f} = \boldsymbol{\Psi}\mathbf{x} \tag{4}$$

Fortunately, the NP-hard problem P_0 is equivalent to the ℓ_1-minimization problem P_1:

$$(P_1) \quad \min \|\mathbf{x}\|_{\ell_1} \quad s.t.\, \mathbf{y} = \boldsymbol{\Phi}\mathbf{f},\, \mathbf{f} = \boldsymbol{\Psi}\mathbf{x} \tag{5}$$

when the so-called sensing matrix $\mathbf{A} = \boldsymbol{\Phi}\boldsymbol{\Psi} \in \mathbb{R}^{M \times N}$ satisfies the Restricted Isometry Property (RIP) or in most practical scenarios, has low mutual coherence. And problem P_1 can be solved by convex optimization within an acceptable period of time.

It has been proved that if and only if $\boldsymbol{\Phi}$ satisfies the Null Space Property [2], the problem P_0 and P_1 are equivalent. However, the null space property is usually somewhat difficult to show directly, instead, RIP [3], [4] is introduced by Candès and the method of analysing mutual coherence [5] of matrices is developed by Donoho to estimate the performance of sensing matrices.

3. Partial Kronecker Product of Identity and Hadamard Matrix

3.1 The Structure of PKIH

Before we show the definition of the PKIH, we firstly introduce the Kronecker product of Identity and Hadamard matrix (KIH). And the definitions of KIH and PKIH are as follows:

Definition 1: Let $U \in \mathbb{R}^{N \times N}$ and $d, p, q, i \in \mathbb{N}$, and denote $\mathbf{H}_d$ is a $d \times d$ Hadamard matrix, the KIH matrix is defined as

$$\mathbf{U} = \mathbf{I}_p \otimes \left(\mathbf{H}_d \otimes \mathbf{I}_q\right), d = 4i \tag{6}$$

PKIH is then defined as:

Definition 2: Let $\mathbf{I}_N^{\mathfrak{X}} \in \mathbb{R}^{M\times N}$ and $\lambda \in \mathbb{N}$, then the PKIH matrix $\boldsymbol{\Phi} \in \mathbb{R}^{M\times N}$ is defined as

$$\boldsymbol{\Phi} = \mathbf{I}_N^{\mathfrak{X}} \mathbf{U}_N \mathbf{I}_N^{\mathfrak{Z}} \tag{7}$$

where the subset $\mathfrak{X} \subset [N]$ is selected uniformly at random among all subsets of $[N]$ of cardinality $|\mathfrak{X}| = M$, $\mathfrak{Z} = \left(\lambda \cdot [N] \bmod N\right) + 1$ defines a deterministic permutation of $[N] \to [N]$. In matrix representation, $\mathbf{I}_N^{\mathfrak{X}}$ is simply a random subset of M rows of an $N \times N$ identity matrix and $\mathbf{I}_N^{\mathfrak{Z}}$ is a deterministic permutation matrix. In this paper, $\mathbf{I}_N^{\mathfrak{X}}$ is called random down sampling matrix.

According to the structure of PKIH, the matrix can be determined uniquely by p, d, q and M random variables uniformly selected from [1, N]. The number of random variables needed is equal to that of measurements to be generated. The value of all the non-zero entries is in {0, ±1}, namely it is a ternary valued matrix. Moreover, the sparsity of the sensing matrix is d/N and every d operations or flops generate a measurement since there are d non-zero entries in each row of the matrix. During the sampling procedure, every signal point is sampled about p times. And it takes d operations to generate one measurement, therefore, it takes roughly $m = dM$ operation flops to acquire a N points signal by rate M. Thus the matrix is sparse, ternary valued, highly structured, and uses few random numbers. The scatter plot of the matrix is plotted in **Fig. 1** to show the structure of PKIH. And the properties of the PKIH are showed by following lemmas in detail.

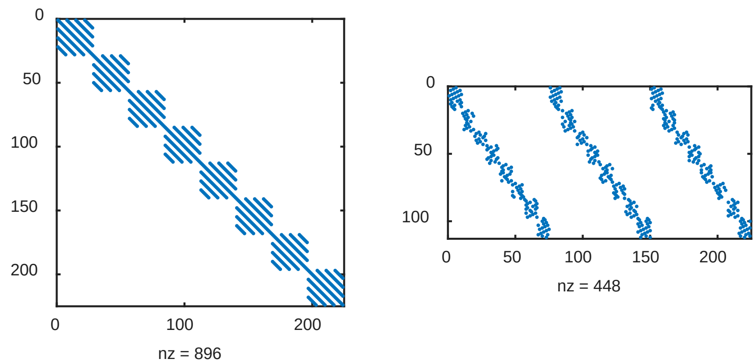

Fig. 1. Scatter plots of KIH (left) and PKIH (right) with p=8, d=4, q=7 and λ=3. The sparsity of the matrix is 0.0179.

Lemma 1: Let $\mathbf{U} \in \mathbb{R}^{N\times N}$ be a KIH matrix, then $\mathbf{U}$ is an orthogonal matrix.

Proof: By the transpose property and the mixed-product property of Kronecker product, we have,

$$\begin{aligned}\mathbf{U}^T\mathbf{U} &= \left(\mathbf{I}_p \otimes \left(\mathbf{H}_d \otimes \mathbf{I}_q\right)\right)^T \left(\mathbf{I}_p \otimes \left(\mathbf{H}_d \otimes \mathbf{I}_q\right)\right) \\ &= \left(\mathbf{I}_p^T \otimes \mathbf{H}_d^T \otimes \mathbf{I}_q^T\right)\left(\mathbf{I}_p \otimes \mathbf{H}_d \otimes \mathbf{I}_q\right) \\ &= \left(\mathbf{I}_p^T\mathbf{I}_p\right)\otimes\left(\mathbf{H}_d^T\mathbf{H}_d\right)\otimes\left(\mathbf{I}_q^T\mathbf{I}_q\right) \\ &= \mathbf{I}_p \otimes \left(d\mathbf{I}_d\right)\otimes \mathbf{I}_q \\ &= d\mathbf{I}_{pdq}\end{aligned} \tag{8}$$

which indicates that $\mathbf{U}$ is an orthogonal matrix.

Lemma 2: Let $\mathbf{\Phi} \in \mathbb{R}^{M\times N}$ be a PKIH matrix of size $M \times N$, then it requires M random numbers to construct a $\mathbf{\Phi}$.

Proof: According to Definition 1, any KIH matrix $\mathbf{U}$ can be uniquely determined by d, p and q, namely $\mathbf{U}_N$ is deterministic thus random numbers is not needed during the matrix construction.

By Definition 2, the subset $\mathfrak{X} \subset [N]$ is selected uniformly at random among all subsets of $[N]$ of cardinality $|\mathfrak{X}| = M$. Therefore, one need M random numbers to determine the index set $\mathfrak{X}$ and further, $\mathbf{I}_N^{\mathfrak{X}}$. On the other hand, $\mathfrak{Z}$ can be uniquely determined by the parameter λ thus it does not need random numbers during the construction of $\mathbf{I}_N^{\mathfrak{Z}}$.

In summary, $\mathbf{\Phi} = \mathbf{I}_N^{\mathfrak{X}}\mathbf{U}_N$ can be determined by d, p, q, λ and $\mathfrak{X}$. It requires $0+|\mathfrak{X}|+0 = M$ random numbers to construct a $\mathbf{\Phi}$.

Lemma 3: Let $\mathbf{\Phi} \in \mathbb{R}^{M\times N}$ be a PKIH matrix of size $M \times N$, and let d be the rank of hadamard matrix used to construct $\mathbf{\Phi}$, then the sparsity of $\mathbf{\Phi}$ is d/N.

Proof: By the definition of KIH, $\mathbf{U}_N = \mathbf{I}_p \otimes \left(\mathbf{H}_d \otimes \mathbf{I}_q\right)$, there are $pd^2q = dN$ nonzero entries in $\mathbf{U}_N$ and there are d nonzero entries in each row of $\mathbf{U}_N$. For PKIH, $\mathbf{\Phi} = \mathbf{I}_N^{\mathfrak{X}}\mathbf{U}_N\mathbf{I}_N^{\mathfrak{Z}} = \mathbf{U}_N^{\mathfrak{X}}\mathbf{I}_N^{\mathfrak{Z}}$, where $\mathbf{U}_N^{\mathfrak{X}}$ actually selects rows of $\mathbf{U}_N$ indexed by set $\mathfrak{X}$. And $\mathbf{I}_N^{\mathfrak{Z}}$ is a permutation matrix which does not change the sparsity of $\mathbf{U}_N^{\mathfrak{X}}$. Therefore, the sparsity of $\mathbf{\Phi}$ is $\dfrac{d|\mathfrak{X}|}{MN} = \dfrac{d}{N}$.

The lemma also implies the high efficiency of the PKIH. It takes only d operations or flops to generate one measurement.

3.2 Incoherence Analysis

By the definition of mutual coherence [5], the mutual coherence of an orthonormal matrix $\mathbf{\Phi} \in \mathbb{R}^{N\times N}$ and another orthonormal matrix $\mathbf{\Psi} \in \mathbb{R}^{N\times N}$ is

$$\mu(\mathbf{\Phi},\mathbf{\Psi}) = \max_{1\le i,j\le N}\left|\left\langle \phi_i, \varphi_j \right\rangle\right| \tag{9}$$

where ϕ_i is the *i-th* row of $\mathbf{\Phi}$ and φ_j is the *j-th* column of $\mathbf{\Psi}$.

Furthermore, the Theorem 1.1 of [3] states that for an $N \times N$ orthogonal matrix $\mathbf{\Lambda}$, fix a subset $\mathfrak{T}$ of the signal domain, choose a subset $\mathfrak{S}$ of the measurement domain of size $|\mathfrak{S}| = M$, and choose a sign sequence $\mathbf{z}$ on $\mathfrak{S}$ uniformly at random. For some fixed numerical constants C and C' , when $M \ge C \cdot |\mathfrak{T}| \cdot \mu^2(\mathbf{\Lambda}) \cdot \log(N/\delta)$ and also $M \ge C' \cdot \log^2(N/\delta)$, then with probability exceeding $1-\delta$, every signal $\mathbf{x}$ supported on $\mathfrak{T}$ with signs matching $\mathbf{z}$ can be recovered from $\mathbf{y} = \mathbf{\Lambda}^{\mathfrak{S}}\mathbf{x}$. And based on these results, the theorem of PKIH is the following.

Theorem: Let C, C' and δ be some positive constant and let $\mathbf{\Psi}$ be any orthogonal basis, then with probability at least $1-\delta$, the matrix $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}_{PKIH}\mathbf{\Psi}$ can recover any *K*-sparse signal exactly from $\mathbf{y} = \tilde{\mathbf{\Phi}}\mathbf{x}$ by ℓ_1 minimization if the number of measurements $M \ge \frac{CK}{N}\log(\frac{N}{\delta})$ and also $M \ge C'\log^2(N/\delta)$.

Proof: Let $\mathbf{U} = \mathbf{\Phi}_{KIH}\mathbf{I}^3$ corresponds to $\mathbf{\Phi}_{PKIH}$. In the proof, we will firstly prove that $\mathbf{\Omega} = \mathbf{U}\mathbf{\Psi}$ is an orthogonal matrix and then the mutual coherence of $\mathbf{U}$ and $\mathbf{\Psi}$ is calculated. At last, by applying Theorem 1.1 in [3] to above results, we get the theorem proved.

According to Lemma 1, $\mathbf{\Phi}_{KIH}$ is an orthogonal matrix. Knowing that $\mathbf{I}^3$ is a permutation matrix and $\mathbf{\Psi}$ is an orthogonal basis, we have,

$$\begin{aligned} \mathbf{\Omega}^T\mathbf{\Omega} &= \left(\mathbf{\Phi}_{KIH}\mathbf{I}^3\mathbf{\Psi}\right)^T\left(\mathbf{\Phi}_{KIH}\mathbf{I}^3\mathbf{\Psi}\right) \\ &= \mathbf{\Psi}^T\left(\mathbf{I}^3\right)^T\mathbf{\Phi}_{KIH}^T\mathbf{\Phi}_{KIH}\mathbf{I}^3\mathbf{\Psi} \\ &= d\mathbf{I} \end{aligned} \tag{10}$$

where d is the rank of the Hadamard matrix used to construct $\mathbf{\Phi}_{KIH}$. Therefore, $\mathbf{\Omega}$ is an orthogonal matrix.

Considering the mutual coherence of $\mathbf{U}$ and $\mathbf{\Psi}$, we have,

$$\mu(\mathbf{U},\mathbf{\Psi}) = \max_{\substack{1\le k\le N \\ i=\lambda k \bmod N}} \sum_{j=1}^{N} \mu_{i,j}\varphi_{j,i} \le \max_{1\le i\le N} \sum_{u_{i,j}\ne 0} \varphi_{j,i} = \frac{cd}{\sqrt{N}} \tag{11}$$

where λ is a positive integer and c is some positive constant. For example, $c=\sqrt{2}$ when the basis is DCT basis, and $c=\sqrt{|s|}$ when the basis is Wavelet basis, where s is the width of the Wavelet subband.

By Definition 2,

$$\tilde{\mathbf{\Phi}} = \mathbf{\Phi}_{PKIH}\mathbf{\Psi} = \mathbf{I}_N^{\mathfrak{x}}\mathbf{U}_N\mathbf{\Psi} = \mathbf{U}_N^{\mathfrak{x}}\mathbf{\Psi} = \mathbf{\Omega}^{\mathfrak{x}} \tag{12}$$

where the subset $\mathbf{I}_N^{\mathfrak{x}}$ is a random down sampling matrix thus $\tilde{\mathbf{\Phi}}$ can be viewed as a random down sampled partial matrix of orthogonal matrix $\mathbf{\Omega}$. Therefore, by introducing above results to Theorem 1.1 in [3] we have when,

$$M \geq C_0 \cdot |\Im| \cdot \mu^2(\tilde{\Phi}) \cdot \log(N/\delta) = C_0 K \frac{(cd)^2}{N} \log(N/\delta) = \frac{CK}{N} \log(N/\delta) \quad (13)$$

and also $M \geq C' \log^2(N/\delta)$, the theorem holds. Where C, C' and δ are some positive constants.

This bound is weak and the performance of the matrix behaves better than the theorem holds. As well known, the Gaussian dense matrix, whose entries are normally distributed, has near optimal performance. As a glimpse of the sensing performance of PKIH, we find that entries of $\mathbf{\Phi\Psi}$ are asymptotically normally distributed. As a conjecture, we claim that the sensing performance of PKIH is close to the optimal one. It is firstly illustrated in **Fig. 2**, which depicts the histograms and the fitted curves from the normal distribution of entries of $\mathbf{\Phi\Psi}$ using the 'histfit' function of MATLAB, where $\mathbf{\Psi}$ is the 512×512 DCT matrix and $\mathbf{\Phi}$ is chosen as one of the instances of SRMG, SRML or PKIH matrix of size 256×512. As clearly shown in the figure, the histogram of entries of $\mathbf{\Phi}_{SRMG}\mathbf{\Psi}$ matches the fitted curve of the normal distribution perfectly and that of $\mathbf{\Phi}_{PKIH}\mathbf{\Psi}$ matches well of its fitted curve except a spark at zero, which implies that the matrix contains more zeros. Meanwhile, the histogram of $\mathbf{\Phi}_{SRML}\mathbf{\Psi}$ is reluctant to comply with the fitted curve of normal distribution with some burrs and a spark at zero. Among these curves, the histogram of SRMG model shows best agreement with its fitted curve of normal distribution while the histogram of SRML model shows the worst agreement with its fitted curve of normal distribution. This implies that the sensing performance of PKIH may be better than that of SRML and is close to that of SRMG. Corroborations of this conjecture are provided by simulation results in section 3.

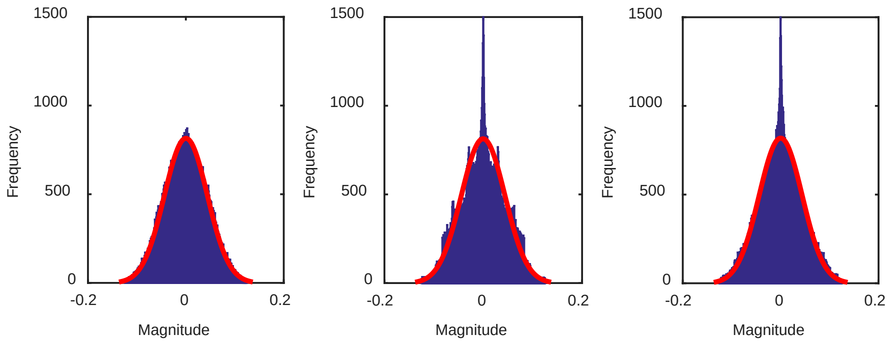

Fig. 2. Histfit of entries of $\Phi_{256\times512}\Psi_{DCT512}$ versus standard normal distribution. The models are SRM-G (left), SRM-L (middle) and PKIH with q=5, λ=3 (right).

3.3 Fast sampling with PKIH

For large signal acquiring, constructing a KIH matrix and subsampling are unsubstantial. According to the structure and the properties of PKIH, it is a highly sparse sign matrix which only has a few entries of $\{\pm 1\}$. The corresponding KIH matrix is highly structured and the permutation matrix is a deterministic matrix. Taking advantage of these natures, we proposed a fast sampling algorithm to implement the sampling procedure of PKIH in a real-time way. Besides, the sampling algorithm can easily be parallelized and distributed in practice.

The sampling procedure using PKIH can be described as follows:

Step 1: Initialize the random seed. Get the input variables of signal length N, rank of

Hadamard matrix d , measurements number M and other parameters including p , q and λ . The average sampling count $r = \left\lceil dq \cdot \frac{M}{N} \right\rceil = \left\lceil \frac{M}{p} \right\rceil$.

Step 2: Construct a $d \times d$ Sylvester-Hadamard matrix $\mathbf{H}_d$. Calculate $\tilde{\mathbf{H}}_{dq}$ by $\tilde{\mathbf{H}}_{dq} = \mathbf{H}_d \otimes \mathbf{I}_q^{\{1\}}$, where $\mathbf{I}_q^{\{1\}}$ is the first row of $\mathbf{I}_q$. Arrange the signal vector $\mathbf{x}$ by $x_i = x_{1+\lambda i \bmod N}, i \in [N]$. Read signal of length dq into memory.

Step 3: Let $\Theta = dq$, randomly and uniformly select a number $\theta \in [\Theta]$ as row index and $\tilde{\mathbf{H}}_{dq}^{\{\theta\}}$ indicates the θ-th row of $\tilde{\mathbf{H}}_{dq}$. Let $\mathbf{h}$ be the index vector of nonzero entries of $\tilde{\mathbf{H}}_{dq}^{\{\theta\}}$ and $\mathbf{h}^-$ be the index vector of negative entries of $\tilde{\mathbf{H}}_{dq}^{\{\theta\}}$, let $\mathbf{x}(\mathbf{h}^-) = -\mathbf{x}(\mathbf{h}^-)$, then $m = \sum \mathbf{x}(\mathbf{h})$ is the generated measurement. Repeat this step for r times to get r measurements.

Step 4: Load a new slice of signal of length dq and return to step 3. Padding zeros when the signal length is not enough, and finish sampling when there's no remain signal.

During the sampling procedure, every signal point is sampled about $\lceil M/p \rceil$ times. And it takes about d operations to generate one measurement because there are only d non-zero entries in every row of the sensing matrix, therefore, it takes roughly $m = dM$ operation flops to acquire a signal of length N by rate M . The fast sampling method is memory efficient and can be easily parallelized. It needs a length of roughly $(d^2 + d)q$ memories for any signal sampling. And this memory requirement can be optimized to roughly $d^2 + dq$. Note that the sampling process of PKIH occupies no multiplication/division operations thus results in easier hardware implementation and faster processing speed in practice.

4. Simulations and Analysis

Several simulations have been taken in order to evaluate the sensing performance and computational complexity of proposed matrix. All the evaluations were executed on MATLAB 2010b on a desktop with 3.0 GHz AMD X4 640 CPU and 4GB RAM. And the structurally random matrices with local model (SRML) and global model (SRMG) in [23], which are already of low complexity and have near optimal sensing performance, were taken into comparison.

Table 1. Practical feature comparison

Features	**PKIH**	**SRMG**	**SRML**
Sparsity	*d/N*	*d/N*	*d/N*
Required Random variables	*M*	*M+N*	*M+N*
Main operations	'+', '-'	'+', '-', '×'	'+', '-', '×'
Memory consuming	*Little more than dq*	*Little more than N*	*Little more than d*
Parallelization	Yes	After pre-randomization	Yes

Firstly, the complexity features of PKIH and SRM models are compared in **Table 1**. Both of them are very sparse, which lead to low computational complexity. But PKIH requires much fewer random variables, low memory costs and no multiplication operations, which lead to easier hardware implementation and higher processing speed in practice. Note that based on the fact that the PKIH saves at least 50% of random variables than that of SRM. This brings improvement of computation and hardware implementation.

4.1 Simulation With Sparse Signals

In this section, the sensing performance of PKIH is evaluated and is compared with that of the completely random projection and also with that of the structurally random matrices in [23], which are already of low complexity. Then, the computational complexity of constructing a sensing matrix is evaluated by counting the CPU time.

Simulation 1: In the first simulation, the input signal $\mathbf{x}$ of length $N=512$ is sparse in the DCT domain with $\mathbf{x}=\mathbf{\Psi\theta}$, where the sparsifying basis $\mathbf{\Psi}$ is the 512×512 IDCT matrix and the transform coefficient vector $\mathbf{\theta}$ has $K=30$ nonzero entries whose magnitudes are Gaussian distributed and locations are at uniformly random. With the signal $\mathbf{x}$, the measurement vectors of length M (varies from 70 to 160) are generated by $\mathbf{y}=\mathbf{\Phi x}$, where $\mathbf{\Phi}$ can be PKIH matrices ($q=5$, $\lambda=3$, PKIH(5,3)), SRMG, SRML, or completely Gaussian random matrices (GAU) [4]. In addition, the block sizes of the sub matrix of the former three models are set to 4. The OMP [24] algorithm is taken as recover strategy and the exact recovery means that the recovered signal $\hat{\mathbf{x}}$ and the original signal $\mathbf{x}$ satisfy the condition of $PSNR=10\log_{10}\left(\|\mathbf{x}\|_2^2/\|\mathbf{x}-\hat{\mathbf{x}}\|_2^2\right)\geq 50$. Every experiment is repeated 10000 times and the probability of exact recovery is showed in **Fig. 3**.

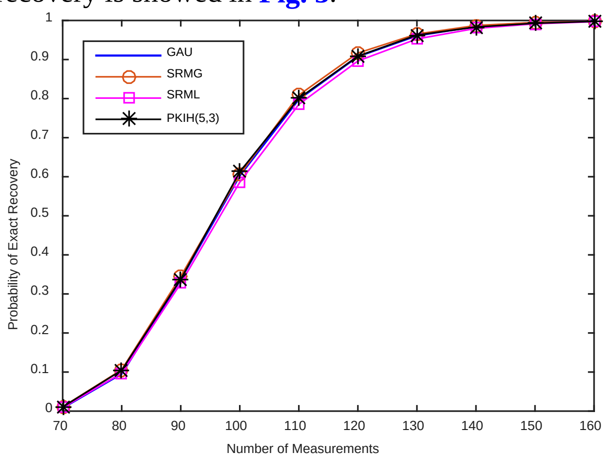

Fig. 3. The probability of exact recovery of different sensing matrices in terms of exact 30 sparse signal of length 512

As illustrated in **Fig. 3**, for a signal of length 512 with 30 randomly located spikes, the probability of exact recovery increases along with the increasing of the number of the measurements generated by each sensing matrix. All these models have nearly the same performance of recovering exactly sparse signals. In detail, the PKIH matrix and the SRMG matrix have almost the same performance and both of them are identical or even a little superior to that of GAU matrix. On the other hand, the curve of the probability of exact recovery of SRML matrix is slightly below that of GAU matrix. This result coincides with the concept of **Fig. 2**. The distributions of the entries of $\mathbf{\Phi\Psi}$ of SRMG and PKIH are closer to normal distribution than that of SRML and result in higher performance, which proves the conjecture illustrated in subsection 2.3.

Above all, based on the fact that $M << N$, constructing a PKIH matrix saves at least 50% of random numbers than that of SRM matrix, which uses $M + N$ random numbers during matrix construction. These results indicate that the PKIH model could achieve the same performance of the SRM models and Gaussian random matrices with much less random numbers.

Simulation 2: In this simulation, the CPU time occupied by constructing the sensing matrices $\mathbf{\Phi}$ corresponding to Simulation 1 is evaluated. All the parameters of these models are unchanged except the sampling rate is set to 0.5 and the signal length corresponding to $\mathbf{\Phi}$ is varied from 500 to 5000 for simplicity. And for lager matrices, one may need to apply some kind of piecewise computing algorithm in order to avoid running out of memory, which is demonstrated in the next section. The Gaussian random matrix is not compared because of its known high complexity. In each experiment we count the CPU time of generating 100 instances of the specified matrix. Every experiment is repeated 100 times and the average CPU time is showed in **Fig. 4**.

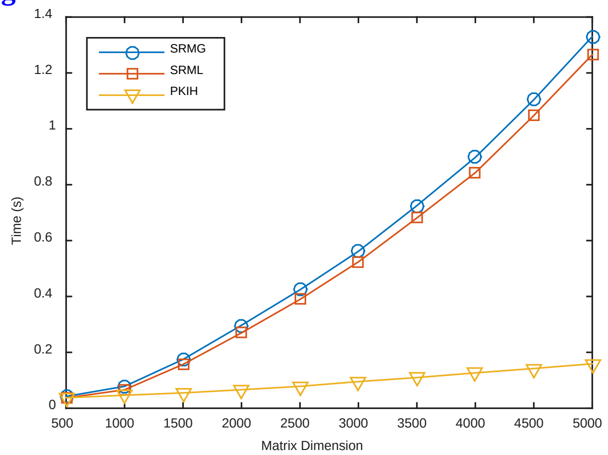

Fig. 4. The CPU time occupied by constructing a sensing matrices versus dimensions of the sensing matrices

As clearly seen in **Fig. 4**, constructing a PKIH matrix consumes the least CPU time of all and it costs a little less time of generating a SRML matrix than that of a SRMG matrix. Moreover, the time costs of both SRMG and SRML increase rapidly along the increasing of matrix dimensions while the PKIH has a flat curve of that. These results indicate that by decreasing the using of random numbers, which usually has a time consuming generating algorithm, the PKIH model significantly reduces the computational complexity of sensing matrix construction.

4.2 Simulation With Compressible Signals

To evaluate the sensing performance for large and compressible signals, the rate-distortion (R-D) performance for standard test images is simulated. Signals of interest are natural images at the resolution of 256×256, including Lena, Cameraman, Peppers, and Boats images. The well-known Daubechies 9/7 wavelet transform is used as the sparsifying basis $\mathbf{\Psi}$ and all images are implicitly regarded as 1-D signals of length 256^2. The GPSR [25] algorithm is taken as recover strategy and the sampling rate ranges from 0.1 to 0.5.

For such a large scale simulation, it takes a huge amount of system resources to implement the sensing method of a completely random matrix. Thus, for the purpose of benchmarking, a more practical scheme of partial FFT [7] in the wavelet domain (WPFFT) is adopted. The WPFFT is to sense wavelet coefficients in the wavelet domain using the method of partial FFT. Theoretically, WPFFT has optimal performance as the Fourier matrix and is completely incoherent with the identity matrix. Note that the WPFFT still requires substantial amount of system resources because it is dense and the sensing procedure is applied in transform domain.

Simulation 3: In this simulation, the sensing matrix $\mathbf{\Phi}$ is chosen as PKIH matrices ($q = 39$, $\lambda = 37$, PKIH(39,37)), SRM matrices of global model (SRMG), SRM matrices of local model (SRML), or WPFFT matrices [4]. The block size of the sub matrix of the former three models is set to 32. Besides, the SRML matrix with block size of sub matrix of 512 (SRML512) is taken into comparison. For each model, the corresponding fast computing algorithm is applied and every experiment is repeated 50 times. The curves of average reconstructed PSNR of these sensing matrices are showed in **Fig. 5**(a), (b), (c), and (d), which correspond to the input Lena, Cameraman, Peppers, and Boats images, respectively.

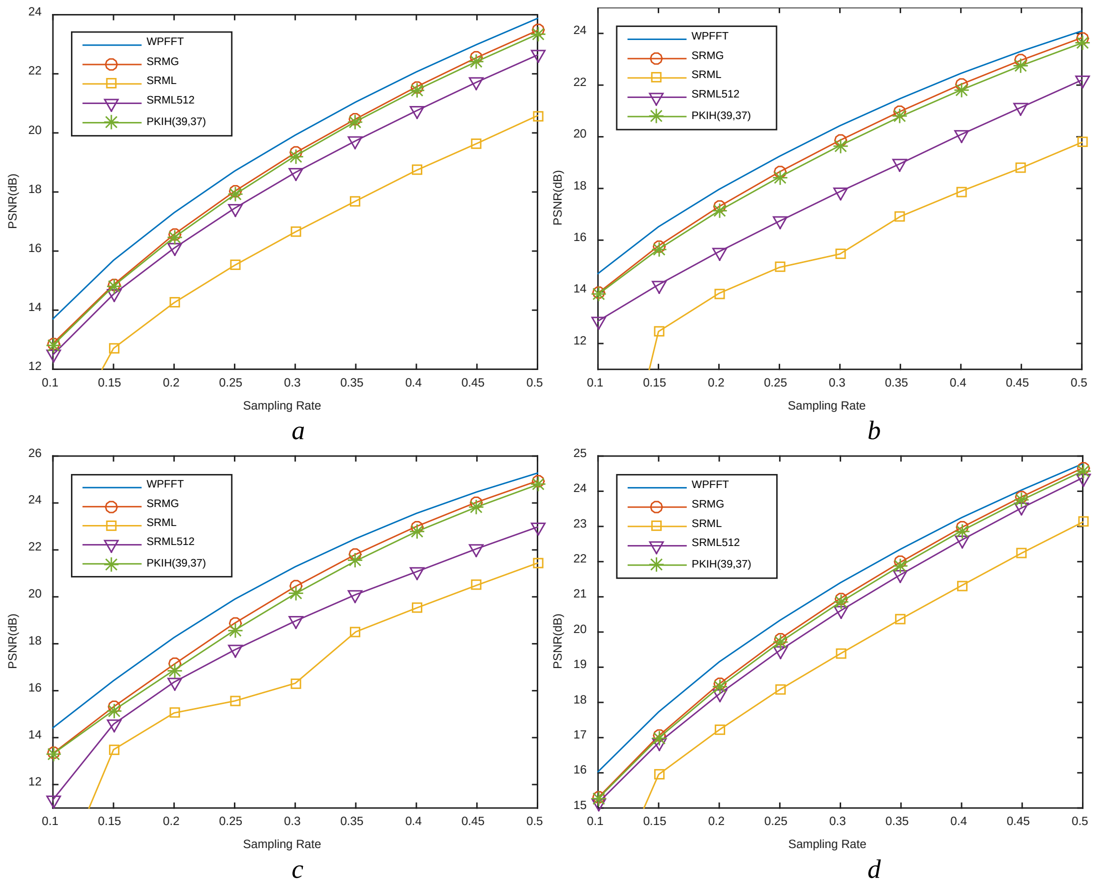

Fig. 5. Performance curves: Quality of signal reconstruction versus sampling rate M/N. (a) The 256×256 Lena image. (b) The 256×256 Cameraman image.(c) The 256×256 Peppers image. (d) The 256×256 Boats image.

There are a few notable observations from these experimental results. Firstly, the SMRL matrices are not efficient enough for sensing smooth signals like images. And by increasing the number of nonzero entries, the SRML512 matrix has got stable sensing performance but still not good enough. On the other hand, the PSNR curves of both the PKIH matrix and the

SRMG matrix are close to that of WPFFT, which is theoretically optimal. Meanwhile, there is almost no observable difference between the two curves. In particular, the maximum PSNR differences between the curves PKIH and SRMG are within 0.2dB in all cases. Besides, comparing with SRML matrix, the PKIH matrix has performance gain from 1.5dB to at most 4.5dB on average. And the PSNR curves of it outperform that of the SRM512 matrix from 0.2dB to 1.5dB on average. Note that the number of nonzero entries in a SRM512 matrix is 16 times more than that in a PKIH matrix.

Above all, the number of random numbers used by PKIH is 9%-33% of that used by SRMG or SRML. This implies that the well-designed sensing structure is an efficient alternative of the randomness of sensing matrices.

Simulation 4: In this simulation, the CPU time of sensing a large scale signal using fast computing algorithm corresponding to sensing matrices $\mathbf{\Phi}$ of Simulation 3 is evaluated. All the parameters of these models remains unchanged except the sampling rate is set to 0.5 and the length of the input signal varies from 5000 to 5×10^5. Every experiment is repeated 1000 times and the average CPU time is plotted in **Fig. 6** using logarithmic coordinates.

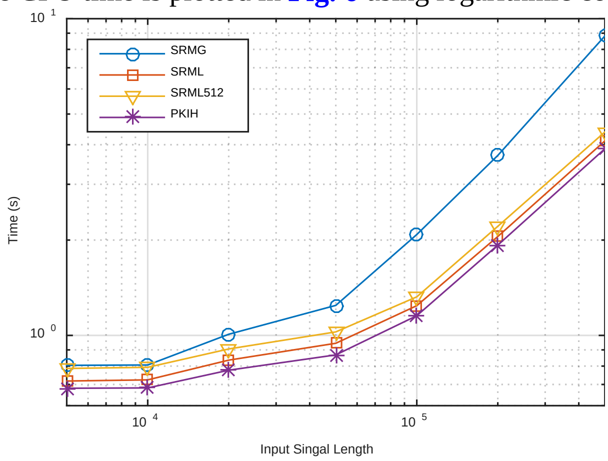

Fig. 6. The processing time of sensing the input signal using specific sensing matrix versus the length of the signal

Clearly shown in the figure, the lines from top to bottom are curves of SRMG, SRML512, SRML and PKIH, which are corresponding to the CPU time occupied by these matrices along the increasing of the input signal length. The SRMG matrix costs the longest time and the cost increases more quickly than the others. That is because the sensing procedure of the SRMG matrix not only needs a large amount of random numbers, but also needs to determine the signal positions participating in the operation uniformly by the random numbers. On the other side, the rising tendency of SRML, SRML512 and PKIH appear the same. Meanwhile, the SRML costs more time than that of PKIH because of using more random numbers and the SRML512 costs more time than that of SRML because of it contains more nonzero entries. In addition, in terms of sensing signals of length $256^2 \approx 6.5\times10^4$, PKIH takes roughly 65% of the time used by SRMG and 85% of that used by SRML512. In a word, the PKIH matrix achieves lowest computational complexity of sensing large signals.

5. Conclusion

Sensing matrix design is one of three main topics of compressive sensing. Dense sensing matrix is computational costly while pure random sensing matrix is hard to implement. This paper developed a new kind of sensing matrix along with the corresponding fast sampling algorithm for large signal acquiring. The matrix is sparse, highly structured and the required number of random numbers is equal to that of generated measurements. In practical applications, these natures bring great benefits to parallelization, real-time processing, and easy hardware implementation. Simulation results showed that, comparing with SRMG, the proposed semi-deterministic matrices maintained almost the same sensing performance using 9%-33% random numbers and 65% sensing time. And it achieves 0.2dB to 1.5dB performance gain on average and uses 85% sensing time in terms of sensing nature images in comparison with SRML512. These results indicate that the proposed semi-deterministic matrix significantly reduced the randomness and the computational complexity while maintained almost the same sensing performance with the popular sampling matrices. However, efficient recover algorithm of proposed matrix is not developed. Fortunately, in terms of applications using compressive sensing, the computational burden is at the decode side. And we will explore fast streaming decoding algorithm like ℓ_1-Homotopy [26] in our future work.

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